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CLASSIFICATION OF THE ISOMORPHIC TYPES OF MARTINGALE-H^1 SPACES

BY

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ABSTRACT

Let $(\mathscr{F}_n)$ be an increasing sequence of finite fields on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ where $\mathscr{F}$ denotes the σ-algebra generated by $\bigcup \mathscr{F}_n$. Then $H^1[(\mathscr{F}_n)]$ is isomorphic to one of the following spaces: $H^1(\delta)$, $(\Sigma H_n^1)_{l^1}$, l^1.

Introduction

In his paper [4] B. Maurey asks: "Peut on classifier les classes d'isomorphism des espaces $H^1[(F_n)]$?" In this note we show that such a classification is indeed possible.

More precisely we have the following

THEOREM 1. *Let $H^1[(\mathscr{F}_n)]$ be infinite dimensional:*

(a) *If l^2 embeds into $H^1[(\mathscr{F}_n)]$ then $H^1[(\mathscr{F}_n)]$ is isomorphic to $H^1(\delta)$.*

(b) *If l^2 does not embed into $H^1[(\mathscr{F}_n)]$ and if $H^1[(\mathscr{F}_n)]$ is not isomorphic to a complemented subspace of l^1 then $H^1[(\mathscr{F}_n)]$ is isomorphic to $(\Sigma H_n^1)_{l^1}$.*

(c) *If l^2 does not embed into $H^1[(\mathscr{F}_n)]$ and if $H^1[(\mathscr{F}_n)]$ is isomorphic to a complemented subspace of l^1 then $H^1[(\mathscr{F}_n)]$ is isomorphic to l^1.*

Part (a) of Theorem 1 was proven by the present author in [5]. Part (c) of Theorem 1 holds for any infinite dimensional complemented subspace of l_1 (cf. [3]). The rest of the paper is used to prove part (b). Our method of proof permits at the same time a characterisation of the isomorphic type of given

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$H^1[(\mathscr{F}_n)]$ space in terms of the underlying measure space $(\Omega,(\mathscr{F}_n),\mathbf{P})$ (cf. Theorem 1(a)).

The constructions given below rely on a result taken from Maurey's paper on H^1 spaces. Let's mention two isomorphic invariants which are shared by $H^1(\delta)$, $(\Sigma H_n^1)_{l^1}$ and l^1.

COROLLARY 2. (a) *$H^1[(\mathscr{F}_n)]$ has an unconditional basis;*

(b) *$H^1[(\mathscr{F}_n)]$ is primary (i.e., for any projection P on $H^1[(\mathscr{F}_n)]$ either $P(H^1[(\mathscr{F}_n)])$ or $(\mathrm{Id}-P)(H[(\mathscr{F}_n)])$ is isomorphic to $H^1[(\mathscr{F}_n)]$).*

PROOF. ad(a) By Theorem 1 it is sufficient to observe that $H^1(\delta)$, $(\Sigma H_n^1)_{l^1}$ and l^1 have unconditional basis.

ad(b) $H^1(\delta)$, $(\Sigma H_n^1)_{l^1}$ are primary [6], l^1 is primary; cf. [3].

The Banach space decomposition principle of Pelczynski is repeatedly applied below. It would be very satisfying to construct an unconditional basis in $H^1[(\mathscr{F}_n)]$ and to explicitly analyse properties of such a basis.

§0. Definitions and notations

Let $(\mathscr{F}_n)$ be a sequence of increasing finite fields of subsets of Ω. Let $\mathbf{P}$ be a probability measure on $(\Omega,\mathscr{F})$, where $\mathscr{F}=\bigvee_{n=1}^{\infty}\mathscr{F}_n$. Given $f\in L^1[(\Omega,\mathscr{F},\mathbf{P})]$ we write

$$S(f)(t):=\left(\sum(\mathbf{E}(f\mid\mathscr{F}_n)-\mathbf{E}(f\mid\mathscr{F}_{n-1}))^2\right)^{1/2}(t),$$

$$H^1[(\mathscr{F}_n)]:=\{f\in L^1[(\Omega,\mathscr{F},\mathbf{P})]:S(f)\in L^1[(\Omega,\mathscr{F},\mathbf{P})]\},$$

$$\mathrm{BMO}[(\mathscr{F}_n)]$$

$$:=\left\{f\in L^2[(\Omega,\mathscr{F},\mathbf{P})]:\sup_n\|\mathbf{E}((f-\mathbf{E}(f\mid\mathscr{F}_{n-1}))^2\mid\mathscr{F}_n\|_\infty^{1/2}<\infty\right\}.$$

EXAMPLES. Let $\mathscr{L}_n$ denote the algebra of subsets of $(0,1]$ generated by dyadic intervals of length 2^{-n}.

(1) $H^1[(\mathscr{L}_n)]$ will be called "the dyadic H^1" and denoted by $H^1(\delta)$.

(2) $\mathscr{F}_m:=\mathscr{L}_{\max(n,m)}$, $m\in\mathbf{N}$.

$H^1[(\mathscr{F}_n)]$ will be denoted by H_n^1.

For a different description of these spaces see, e.g., [1], [4], [5].

0. a. *An algebraic basis of* $\mathscr{D}_n := \{f : f$ *is* $\mathscr{F}_{n+1}$ *measurable and* $\mathbf{E}(f \mid \mathscr{F}_n) = 0\}$

Let $l(n)$ denote the numbers of atoms in $\mathscr{F}_n$. Let $\mathscr{A}_n = \{A_{n,k} : 1 \leqq k \leqq l(n)\}$ denote the collection of atoms in $\mathscr{F}_n$. For each $n \in \mathbf{N}$ and $k \leqq l(n)$ we define (n_k) as follows:

$$A_{n,k} = \bigcup_{l=n_{k-1}+1}^{n_k} A_{n+1,l}.$$

We assume that the enumeration of the atoms in $\mathscr{F}_{n+1}$ is such that there are n_k's as above and such that:

$$\mathbf{P}(A_{n+1,j}) \leqq \mathbf{P}(A_{n+1,j+1}) \qquad \text{for } n_{k-1} + 1 \leqq j < n_k.$$

Now we define:

$$h_{n,j} := \begin{cases} 1 & \text{on } A_{n+1,j}, \\ -\dfrac{\mathbf{P}(A_{n+1,j})}{\mathbf{P}(A_{n+1,j+1})} & \text{on } A_{n+1,j+1}. \end{cases}$$

$$\mathscr{E}_n(A_{n,k}) := \{A_{n+1,j} : n_{k-1} + 1 \leqq j < n_k\},$$

$$\mathscr{E}_n := \bigcup_{k=1}^{l(n)} \mathscr{E}_n(A_{n,k}), \quad \mathscr{E} := \bigcup_{n=1}^{\infty} \mathscr{E}_n, \quad E_n := \bigcup_{E \in \mathscr{E}_n} E.$$

The function $h_{n,i}$ may also be indexed by elements of $\mathscr{E}$:

$$h_A := h_{n,j} \qquad \text{iff } A = \{t : h_{n,j}(t) = 1\}.$$

Some comments are in order: For $j := n_k$ the function $h_{n,j}$ is *not* defined. The atom A_{n+1,n_k} is the biggest atom in $\mathscr{F}_{n+1}$ which is a subset of $A_{n,k}$.

$\mathscr{E}_n(A_{n,k})$ contains all atoms of $\mathscr{F}_{n+1}$ which are subsets of $A_{n,k}$ with the exception of A_{n+1,n_k}. $\mathscr{E}$ has the following property: $E \in \mathscr{E}$, $F \in \mathscr{E}$ and $E \cap F \neq 0$ then either $E \subset F$ or $F \subset E$.

Hence, for $\mathscr{G} \subset \mathscr{E}$ we may talk about the maximal subsets of $\mathscr{G}$ with respect to inclusion.

We use the following notations below:

For $J \subset \Omega$ we define:

$$G_1(J) := \{E \in \mathscr{E}, E \subseteq J, E \text{ maximal}\} \quad \text{and} \quad G_n(J) := \bigcup_{I \in G_{n-1}(J)} G_1(I).$$

Let $\mathscr{D}$ be a collection of subsets of Ω. For $J \subset \Omega$ we write:

$$J \cap \mathscr{D} := \{D : D \in \mathscr{D} \text{ and } D \subset J\},$$

$$\mathscr{D}^* := \bigcup \{D : D \in \mathscr{D}\}, \quad G_1(J, \mathscr{D}) := G_1(J) \cap \mathscr{D},$$

$$G_n(J, \mathscr{D}) := \bigcup_{I \in G_{n-1}(J, \mathscr{D})} G_1(I, \mathscr{D}).$$

Theorem 1 can be rephrased in terms of the underlying measure space $(\Omega, (\mathscr{F}_n), \mathbf{P})$:

$$A^\infty := \bigcap_{\varepsilon > 0} \bigcup \{B : B \in \mathscr{A}_l, l \in \mathbf{N}, \mathbf{P}(B) \leqq \varepsilon\}.$$

THEOREM 1a. *Let $H^1[(\mathscr{F}_n)]$ be infinite dimensional:*
(a) $\mathbf{P}(A^\infty) > 0$ *iff $H^1[(\mathscr{F}_n)]$ is isomorphic to $H^1(\delta)$.*
(b) $\mathbf{P}(A^\infty) = 0$ *and*

$$\sup_{B \in \mathscr{E}} \frac{1}{\mathbf{P}(B)} \sum_{E \in B \cap \mathscr{E}} \mathbf{P}(E) = \infty$$

iff $H^1[(\mathscr{F}_n)]$ is isomorphic to $(\Sigma H_n^1)_{l^1}$.
(c) $\mathbf{P}(A^\infty) = 0$ *and*

$$\sup_{B \in \mathscr{E}} \frac{1}{\mathbf{P}(B)} \sum_{E \in B \cap \mathscr{E}} \mathbf{P}(E) < \infty$$

iff $H^1[(\mathscr{F}_n)]$ is isomorphic to l^1.

Our proof makes use of

THEOREM A (Maurey [4]). *$H^1[(\mathscr{F}_n)]$ is isomorphic to a complemented subspace of $H^1(\delta)$ (provided $(\mathscr{F}_n)$ is an increasing sequence of finite (!) fields).*

THEOREM B ([5]). (a) $\mathbf{P}(A^\infty) > 0$ *implies $H^1[(\mathscr{F}_n)] \cong H^1(\delta)$.*
(b) $\mathbf{P}(A^\infty) = 0$ *implies that l^2 is not isomorphic to a subspace of $H^1[(\mathscr{F}_n)]$.*

§1. Let's first collect a few lemmas, concerning the behaviour of (h_{nj}).

LEMMA 2. $\mathscr{D}_n := \{f : E(f \mid \mathscr{F}_{n+1}) = f \text{ and } E(f \mid \mathscr{F}_n) = 0\}$.

(a) *$\{h_A, A \in \mathscr{E}_n\}$ forms an algebraic basis of $\mathscr{D}_n$.*

(b) *Given $(f_m)_{m\in\mathbf{N}}$ where $f_m \in \mathscr{D}_m$. Then*

$$\int_\Omega \left(\sum_{m=n}^{\infty} f_m^2\right)^{1/2} d\mathbf{P} \leqq 2 \int_{\cup_{m=n}^{\infty} E_m} \left(\sum_{m=n}^{\infty} f_m^2\right)^{1/2} d\mathbf{P}.$$

PROOF. (a) is clear.

(b) The proof is divided into two parts: We first find a minorization of

$$\int_{\cup_{m=n}^{\infty} E_m} \left(\sum_{m=n}^{\infty} f_m^2\right)^{1/2}.$$

This will be followed by a proper majorization of

$$\int_{\Omega \setminus \cup_{m=n}^{\infty} E_m} \left(\sum f_m^2\right)^{1/2}. \qquad \square$$

In both parts the following sets must be studied:

$$\Delta(n,k) := E_n \cup \cdots \cup E_{n+k} \setminus E_n \cup \cdots \cup E_{n+k-1},$$

$$J_{n,k} := \{j : A_{n+k,j} \cap \Delta(n,k) \neq \varnothing\},$$

$$K_{n,k} := \{(n+k)_j - 1 : j \in J_{n,k}\}.$$

Part 1. By (a) f_{n+k} has a well-defined expansion with respect to $(h_{n+k,j})$. Hence there exists a well-defined sequence $(a_{n+k,j})$ *such that* $f_{n+k} = \sum h_{n+k,j} a_{n+k,j}$. Fix $l \in J_{n,k}$. We put $l_1 := ((n+k)_{l-1} + 1)$ and

$$l_2 := (((n+k)_l) - 1).$$

Then

$$(f_{n+k})\chi_{\Delta(n,k)} = \left(\sum_{l \in J_{n,k}} \sum_{j=l_1}^{l_2} a_{n+k,j} h_{n+k,j}\right) \chi_{\Delta(n,k)}.$$

Moreover, by definition of $h_{n+k,j}$:

$$\left(\sum_{j=l_1}^{l_2} a_{n+k,j} h_{n+k,j}\right) \cdot \chi_{\Delta(n,k)}$$

$$= a_{n+k,l_1} \chi_{A_{n+k+1,l_1}} + \sum_{j=l_1+1}^{l_2} \chi_{A_{n+k+1,j}} \cdot \left(a_{n+k,j} - a_{n+k+1,j-1} \cdot \frac{P(A_{n+k+1,j-1})}{P(A_{n+k+1,j})}\right).$$

Hence

$$\int_{\Delta(n,k)} |f_{n+k}| = \sum_{l \in J_{n,k}} |a_{n+k,l_1}| P(A_{n+k+1,l_1})$$

$$+ \sum_{j=l_1+1}^{l_2} P(A_{n+k+1,j}) |a_{n+k,j} - a_{n+k,j-1}| \cdot \frac{P(A_{n+k+1,j-1})}{P(A_{n+k+1,j})}$$

$$\overset{(1)}{\geqq} \sum_{l \in J_{n,k}} |a_{n+k,l_1}| \cdot P(A_{n+k,l_1}) + |a_{n+k,l_2}$$

$$\cdot P(A_{n+k+1,l_2}) - a_{n+k,l_1} \cdot P(A_{n+k+1,l_1})|$$

$$\overset{(2)}{\geqq} \sum_{l \in J_{n,k}} |a_{n+k,l_2}| \cdot P(A_{n+k+1,l_2})$$

$$= \sum_{i \in K_{ni}} |a_{n+k,i}| \cdot P(A_{n+k+1,i}).$$

REMARK. (1) and (2) hold by an application of the triangle inequality.

Now we estimate as follows:

$$\int_{\cup_{m=n}^{\infty} E_m} \left(\sum_{m=n}^{\infty} f_m^2 \right)^{1/2} \geqq \int_{E_n} |f_n| + \sum_{k=1}^{\infty} \int_{\Delta(n,k)} |f_{n+k}|$$

$$\geqq \sum_{k \geqq 0}^{\infty} \sum_{i \in K_{n,k}} |a_{n+k,i}| \mathbf{P}(A_{n+k+1,i}).$$

Part 2. Fix $k \in \mathbf{N}$, $A \in \mathscr{A}_{n+1} \setminus \mathscr{E}_n$. We start with the following identity:

$$f_{n+k} \chi_{(\Omega \setminus \cup_{m>n} E_m) \cap A} = \begin{cases} a_{n+k,i} h_{n+k,i} \cdot \chi_{(\Omega \setminus \cup_{m>n} E_m) \cap A} & \text{if } i \in K_{n,k}, \\ 0 & \text{else.} \end{cases}$$

For A given there exists exactly one $i \in K_{n,k}$ such that $A_{n+k+1,i+1}$ is contained in A. Lets call it $i(k)$. Then we get:

(i) $$A_{n+k+1,i(k)+1} \supset A_{n+(k+1)+1,i(k+1)+1} \supset \cdots.$$

(ii) $$\bigcap_{k>0} A_{n+k+1,i(k)+1} = \left(\Omega \setminus \bigcup_{m \geqq n} E_m \right) \cap A,$$

(iii) $t \in (\Omega \setminus \bigcup_{m \geqq n} E_m) \cap A$ implies

$$h_{n+k,i(k)}(t) = -\frac{\mathbf{P}(A_{n+k+1,i(k)})}{\mathbf{P}(A_{n+k+1,i(k)+1})}.$$

Hence, we have the following identity:

$$\left(\sum_{k\geqq 0} f_{n+k}\right)\chi_{(\Omega\setminus\cup_{m\geqq n}E_m)\cap A}$$

$$=\left(\sum_{\substack{k\geqq 0\\ i\in K_{n,k}}}\ \sum_{A_{n+k+1,i+1}\subset A}(-1)a_{n+k,i}\cdot\frac{\mathbf{P}(A_{n+k+1,i})}{\mathbf{P}(A_{n+k+1,i+1})}\right)\chi_{(\Omega\setminus\cup_{m\geqq n}E_m)}.$$

And this implies:

$$\int_{(\Omega\setminus\cup_{m\geqq n}E_m)\cap A}\left(\sum f_m^2\right)^{1/2}$$

$$=\mathbf{P}\left(\left(\Omega\setminus\bigcup_{m=n}^{\infty}E_m\right)\cap A\right)\left(\sum_{\substack{k\geqq 0\\ i\in K_{n,k}}}\ \sum_{A_{n+k+1,i+1}\subset A}|a_{n+k,i}|^2\cdot\frac{\mathbf{P}^2(A_{n+k+1,i})}{\mathbf{P}^2(A_{n+k+1,i+1})}\right)^{1/2}.$$

We are now thoroughly prepared to understand the following inequalities:

$$\int_{(\Omega\setminus\cup_{m\geqq n}E_m)}\left(\sum_{m\geqq n} f_m^2\right)^{1/2}$$

$$\leqq\sum_{A\in\mathscr{A}_{n+1}\setminus\mathscr{E}_n}\left(\sum_{\substack{k\geqq 0\\ i\in K_{n,k}}}\ \sum_{A_{n+k+1,i+1}\subset A}|a_{n+k,i}|^2\,\mathbf{P}^2(A_{n+k+1,i})\right)^{1/2}.$$

Combining the above estimates we get

$$\int_{\Omega\setminus\cup_{m=n}^{\infty}E_m}\left(\sum f_m^2\right)^{1/2}\leqq\int_{\cup_{m=n}^{\infty}E_m}\left(\sum f_m^2\right)^{1/2}.$$

Hence we get

$$\int\left(\sum_{m=n}^{\infty} f_m^2\right)^{1/2}d\mathbf{P}=\int_{\Omega\setminus\cup_{m=n}^{\infty}E_m}\left(\sum_{m=n}^{\infty} f_m^2\right)^{1/2}d\mathbf{P}+\int_{\cup_{m=n}^{\infty}E_m}\left(\sum_{m=n}^{\infty} f_m^2\right)^{1/2}d\mathbf{P}$$

$$\leqq 2\int_{\cup_{m=n}^{\infty}E_m}\left(\sum_{m=n}^{\infty} f_m^2\right)^{1/2}d\mathbf{P}.$$

Lemma 3. *Suppose that*

$$\sup_{B\in\mathscr{E}} \frac{1}{\mathbf{P}(B)} \sum_{E\in B\cap\mathscr{E}} \mathbf{P}(B) = \infty,$$

then there exists $\mathscr{G} \subset \mathscr{E}$ *such that*:

* *for* $l \in \mathbf{N}$, $E, F \in \mathscr{G} \cap \mathscr{A}_l$ *we get*: $E \cap F = \varnothing$ *implies* $\operatorname{supp} h_E \cap \operatorname{supp} h_F = \varnothing$,

* $\sup_{B\in\mathscr{G}} (1/\mathbf{P}(B)) \sum_{E\in B\cap\mathscr{G}} \mathbf{P}(E) = \infty$.

PROOF. Obvious.

LEMMA 4 (cf. [1] Ch. X, Lemma 3.2). *Given* $\mathscr{B} \subset \mathscr{E}$, $B \in \mathscr{B}$, $n \in \mathbf{N}$, $\gamma < 1$ *such that*

$$\frac{1}{\mathbf{P}(B)} \sum_{E\in B\cap\mathscr{B}} \mathbf{P}(E) > \frac{n}{1-\gamma},$$

then there exists $I \in B \cap \mathscr{B}$ *such that*

$$\Sigma\{\mathbf{P}(A) : A \in G_n(I, \mathscr{B})\} > \gamma \mathbf{P}(I).$$

PROOF. Suppose not; then

$$\begin{aligned}\frac{1}{\mathbf{P}(B)} \sum_{E\in B\cap\mathscr{B}} \mathbf{P}(E) &= \frac{1}{\mathbf{P}(B)} \sum_{m\in\mathbf{N}} \sum_{E\in G_m(B,\mathscr{B})} \mathbf{P}(E) \\ &= \frac{1}{\mathbf{P}(B)} \sum_{i=1}^{n} \sum_{m=\mathbf{N}} \sum_{E\in G_{mn+i}(B,\mathscr{B})} \mathbf{P}(E) \\ &\leqq \frac{1}{\mathbf{P}(B)} \sum_{i=1}^{n} \left(\sum_{m\in\mathbf{N}} \mathbf{P}(B)\gamma^m \right) \\ &\leqq \frac{n}{1-\gamma},\end{aligned}$$

a contradiction!

LEMMA 5. *Let* $\mathscr{A} \subset \mathscr{G}$ *be given*. $\mathscr{G}$ *is as in the conclusion of Lemma* 3.

(a) *For* $h_{\mathscr{A}} := \Sigma\{h_A : A \in \mathscr{A}\}$ *we get*

$$S^2(h_{\mathscr{A}}) = \Sigma\{h_A^2 : A \in \mathscr{A}\}.$$

(b) *There exists* $\mathscr{B} \subset \mathscr{A}$ *such that*

$$\tfrac{1}{2} < S^2(h_{\mathscr{B}})(t), \qquad t \in \mathscr{A}^*;$$

$$\tfrac{3}{2} > S^2(h_{\mathscr{B}})(t), \qquad t \in \Omega.$$

PROOF.

(a)
$$S^2(h_{\mathscr{A}}) = \sum_k \left(\sum \{h_A : A \in \mathscr{A} \cap \mathscr{A}_k\}\right)^2$$

$$= \sum_k \sum \{h_A^2(t) : A \in \mathscr{A} \cap \mathscr{A}_k\}.$$

(b) We will apply a stopping time argument: Define

$$l_0 = \inf\{l : \mathscr{A} \cap \mathscr{A}_l \neq \emptyset\}$$

and put $\mathscr{B} := \mathscr{A} \cap \mathscr{A}_{l_0}$. Next pick $J \in \mathscr{A}_{l_0+1} \cap \mathscr{A}$. We will decide whether or not to put J into our collection $\mathscr{B}$ according to the following rule:

If $S^2(h_{\mathscr{B}})/J > \frac{1}{2}$ then $\mathscr{B}$ remains unchanged.

If $S^2(h_{\mathscr{B}})/J < \frac{1}{2}$ then $\mathscr{B} := \mathscr{B} \cup \{J\}$.

After having played this game with all $J \in \mathscr{A}_{l_0+1} \cap \mathscr{A}$ we consider $J \in \mathscr{A}_{l_0+2} \cap \mathscr{A}$ and continue.

Taking into account that $\bigcup_{l=1}^{\infty} (\mathscr{A}_l \cap \mathscr{A})^* = \mathscr{A}^*$ we arrive at the desired result.

LEMMA 6. *Fix $\mathscr{B} \subset \mathscr{G}$. Fix $n \in \mathbf{N}$. Let $p \in \mathbf{N}$ be the least integer bigger than* $\max(-\ln_2(\frac{1}{2}(1 - 2^{-1/n})), -\ln_2(\frac{1}{2}(2^{+1/n} - 1)))$, *then*

$$\sum \{\mathbf{P}(A) : A \in G_{p \cdot n}(I_0, \mathscr{B})\} \geqq (1 - 8^{-n})\mathbf{P}(I_0)$$

implies that $G_{m \cdot p}(I_0, \mathscr{B})$, $m \leqq n$ may be decomposed into $(\mathscr{B}_{mi})$, $i \in \{0, \ldots, 2^m - 1\}$, such that for $m \leqq n$:

(a) *$I \in \mathscr{B}_{mi}$, $j \in \{0, 1\}$ we get*

$$\mathbf{P}(I \cap \mathscr{B}^*_{m+1,2i+j}) \leqq (\tfrac{1}{2} + 2^{-p})\mathbf{P}(I),$$

(b)
$$\mathscr{B}^*_{m+1,2i} \cap \mathscr{B}^*_{m+1,2i+1} = \emptyset$$

$$\mathscr{B}^*_{m+1,2i} \cup \mathscr{B}^*_{m+1,2i+1} \subset \mathscr{B}^*_{m+1},$$

(c) $\mathbf{P}(I_0)(2^{-m}/2 - 4^{-n}) \leqq \mathbf{P}(\mathscr{B}^*_{m,i}) \leqq \mathbf{P}(I_0)2^{-m} \cdot 2.$

PROOF. We will repeatedly apply the following remark: Given I in $\mathscr{A}_l$, $l \in \mathbf{N}$ then $J \in G_p(I, \mathscr{B})$ implies

$$\mathbf{P}(J) \leqq 2^{-p}\mathbf{P}(I).$$

Step 00. $\mathscr{B}_{0,0} := I_0$. The previous remark gives us: $\mathscr{B}_{1,0}, \mathscr{B}_{1,1} \subset G_p(I_0, \mathscr{B})$ such that for $j \in \{0, 1\}$

$$(\tfrac{1}{2} - 2^{-p})\mathbf{P}(I_0) < \mathbf{P}(\mathscr{B}_{1,j}^* \cap I_0) < (\tfrac{1}{2} + 2^{-p})\mathbf{P}(I_0).$$

Step mj. Suppose that for $m < n$, $\mathscr{B}_{0,0}, \ldots, \mathscr{B}_{m,j}$ are already defined. Pick $J \in \mathscr{B}_{m,j}$ and find $l \in \mathbf{N}$ such that $J \in \mathscr{A}_l$. Applying the remark again we may decompose

$$J \cap G_{mp+p}(I_0, \mathscr{B})$$

into $\mathscr{B}_{m+1,2i+j}(J)$, $j \in \{0, 1\}$ such that

$$(\tfrac{1}{2} - 2^{-p})\mathbf{P}(J \cap G^*_{(m+1)\cdot p}(I_0, \mathscr{B})) \leqq \mathbf{P}(\mathscr{B}^*_{m+1,2i+j}(J)) \leqq (\tfrac{1}{2} + 2^{-p})\mathbf{P}(J \cap G^*_{(m+1)\cdot p}(I_0, \mathscr{B})).$$

Taking the union we obtain the desired decomposition of $\mathscr{B}_{m,i}$, namely:

$$\mathscr{B}_{m+1,2i+j} := \bigcup \{\mathscr{B}_{m+1,2i+j}(J) : J \in \mathscr{B}_{m,i}\}.$$

Taking the sum of the inequalities above we get:

$$\frac{1}{(\tfrac{1}{2} + 2^{-p})} \mathbf{P}(\mathscr{B}^*_{m+1,2i+j}) < \mathbf{P}(\mathscr{B}^*_{m+1} \cap G^*_{(m+1)p}(I_0, \mathscr{B})) \leqq \frac{1}{(\tfrac{1}{2} - 2^{-p})} \mathbf{P}(\mathscr{B}^*_{m+1,2i+j}).$$

Hence

$$\frac{1}{(\tfrac{1}{2} + 2^{-p})} \mathbf{P}(\mathscr{B}^*_{m+1,2i+j}) < \mathbf{P}(\mathscr{B}^*_{m,i}) \leqq \frac{1}{(\tfrac{1}{2} - 2^{-p})} \mathbf{P}(\mathscr{B}^*_{m+1,2i+j}) + 8^{-n}\mathbf{P}(I_0).$$

Now put

$$\alpha = \frac{1}{(\tfrac{1}{2} + 2^{-p})}, \quad \beta = \frac{1}{(\tfrac{1}{2} - 2^{-p})}.$$

Iterating the above procedure we obtain families $(\mathscr{B}_{m,i})$, $m \leqq n$, $i \leqq 2^m - 1$ such that for $j \in \{0, 1\}$

$$\mathcal{B}^*_{m+1,2i} \cap \mathcal{B}^*_{m+1,2i+1} = \varnothing,$$

$$\mathcal{B}^*_{m+1,2i} \cup \mathcal{B}^*_{m+1,2i+1} \subset \mathcal{B}^*_{m,i}$$

and

$$\mathbf{P}(I_0) \geqq \alpha^m \mathbf{P}(\mathcal{B}^*_{m+1,2i+j}),$$

$$\mathbf{P}(I_0) \leqq \beta^m \mathbf{P}(\mathcal{B}^*_{m+1,2i+j}) + (8^{-n})\left(\sum_{k=1}^{m} \beta^k\right).$$

Our choice of p gives now the desired estimates.

LEMMA 7. *Fix $n \in \mathbf{N}$, define p as in Lemma 6 and suppose that there exists $B \in \mathcal{G}$ such that*

$$\frac{1}{P(B)} \sum_{E \in B \cap G} \mathbf{P}(E) \geqq (p \cdot n) \cdot 8^n.$$

Then there exists $I \in B \cap \mathcal{G}$, $\mathcal{D}_{m,j} \subset \mathcal{G} \cap I$, $j \leqq 2^m - 1$, $m < n$ such that:

(a) $i_n : H^1_{n-1} \to H^1[(\mathcal{F}_k)]$, $h_{mj} \to h_{\mathcal{D}_{mj}} \cdot \mathbf{P}(I)^{-1}$ *extends to an isomorphism onto* $\operatorname{span}\{h_{\mathcal{D}_{mj}} : m < n, 0 \leqq j \leqq 2^m - 1\}$.

(b) $P_n : H^1[(\mathcal{F}_k)] \to H^1[(\mathcal{F}_k)]$,

$$f \to \sum_{(mj)} \frac{\langle f, h_{\mathcal{D}_{mj}} \rangle}{\| h_{\mathcal{D}_{mj}} \|_2^2} \cdot h_{\mathcal{D}_{mj}}$$

is a bounded idempotent operator onto $\operatorname{span}\{h_{\mathcal{D}_{mj}} : m < n, 0 \leqq j \leqq 2^m - 1\}$.

PROOF. Lemma 4 implies that there exists $I \subset \mathcal{G}$ such that

$$\sum \{\mathbf{P}(A) : A \in G_{n \cdot p}(I, \mathcal{G})\} > (1 - 8^{-n}) P(I).$$

Hence by Lemma 6 there exists a family $(\mathcal{B}_{m,i})$, $m \leqq n$ having the proposition (a), (b), (c) of Lemma 6.

Next fix $J \in \mathcal{B}_{m,i}$: We apply Lemma 5 to the family $J \cap \{\mathcal{B}_{m+1,2i} \cup \mathcal{B}_{m+1,2i+1}\}$ and denote the resulting subfamily by $\mathcal{D}_{m,i}(J)$.

Finally we put: $\mathcal{D}_{m,i} = \bigcup \{\mathcal{D}_{m,i}(J) : J \in \mathcal{B}_{m,i}\}$ and

$$h_{\mathcal{D}_{m,i}} = \sum \{h_A : A \subset \mathcal{D}_{m,i}\}.$$

To show that i_n extends to an isomorphism we take $(a_{m,i})$, $m < n$, $i \leqq 2^m - 1$ arbitrary. Let's first define $(m, i) \supset (k, j)$ iff $\mathcal{B}^*_{m,i} \supset \mathcal{B}^*_{k,j}$.

$$\left\| i_n\left(\sum a_{m,i}h_{m,i}\right)\right\| = \left\| \sum a_{m,i},\, h_{\mathscr{D}_{m,i}}P(I)^{-1}\right\|$$

$$\overset{(1)}{=} \int \left(\sum a_{m,i}^2 S^2(h_{\mathscr{D}_{m,i}})\mathbf{P}(I)^{-2}\right)^{1/2}$$

$$\overset{(2)}{\geqq} \frac{1}{2}\int \left(\sum_{i=1}^{n} a_{m,i}^2 \chi_{\mathscr{B}^*_{m+1,2i}\cup \mathscr{B}^*_{m+1,2i+1}} \cdot \mathbf{P}(I)^{-1}\right)^{1/2}$$

$$\overset{(3)}{\geqq} \frac{1}{2}\sum_{j=0}^{2^n-1}\left(\sum_{(m,i)\supset(n,j)} a_{m,j}^2\right)^{1/2}\mathbf{P}(I)^{-1}$$

$$\cdot(\mathbf{P}(\mathscr{B}^*_{n+1,2i}) + \mathbf{P}(\mathscr{B}^*_{n+1,2i+1}))$$

$$\overset{(4)}{\geqq} \frac{1}{2}\sum_{j=0}^{2^n-1}\left(\sum_{(m,i)\supset(n,j)} a_{m,j}^2\right)^{1/2}\left(\frac{2^{-n}}{2}-\frac{8^{-n}}{4}\right)$$

$$\overset{(5)}{\geqq} \frac{1}{8}\left\| \sum_{m,i} a_{m,i}h_{m,i}\right\|.$$

(1) — for $i \neq j$: supp $S(h_{\mathscr{D}_{m,i}}) \cap$ supp $S(h_{\mathscr{D}_{m,j}}) = \varnothing$ (this holds because we applied Lemma 5 to the family $J \cap \{\mathscr{B}_{m+1,2i} \cup \mathscr{B}_{m+1,2i+1}\}$ rather than to $J \cap \mathscr{B}_{m,i}$);

— for $m \neq k$ $\mathscr{D}_{m,i}$ and $\mathscr{D}_{m,j}$ are taken from different generations of I_0;

(2) — this is property (b) of Lemma 5;

(3) — properties (b), (c) of Lemma 6;

(4) — property (c) of Lemma 6;

(5) — definition of H_n^1.

It is not difficult to see now that the above chain of inequalities can be reversed (with different constants of course!).

The boundedness of P_n follows from the following fact: For $J \in \mathscr{B}_{m_0 j_0}$ the following holds:

$$h_{\mathscr{D}_{mj}/J} = \text{const} \qquad \text{for } m < m_0,$$

$$\int_J S^2(h_{\mathscr{D}_{mj}}) \leqq \mathbf{P}(J)2^{-m+m_0} \quad \text{for } m \geqq m_0.$$

Now we finish the proof as follows.

As pointed out in [5] P_n is bounded iff there exists $C \in \mathbf{R}^+$ (independent of n) such that for $f = \sum a_{mj}h_{\mathscr{D}_{mj}}$ the following holds:

$$\| f \|^2_{\mathrm{BMO}[(\mathscr{F}_n)]} \leqq C \sup_{(k,i)} 2^i \sum_{(m,j) \subset (k,i)} a^2_{mj} 2^{-j}.$$

To this end, fix $j \in \mathbf{N}$, $I \in \mathscr{A}_j$, $J(\supset I) \in \mathscr{A}_{j-1}$,

$$m_0 := \inf\{m : \exists\, i \leqq 2^m,\ \exists\, E \in \mathscr{G}_{m,i}, E \supset J\},$$

$$j_0 := k \Leftrightarrow J \subset \mathscr{G}^*_{m_0 j}.$$

Observe that for $t \in I$:

$$(f - \mathbf{E}(f \mid \mathscr{F}_{j-1}))(t) = \left(\sum_{m \geqq m_0} a_{mi} h_{\mathscr{G}_{mi}} - \int_J \left(\sum_{m \geqq m_0} a_{mi} h_{\mathscr{G}_{mi}} \right) \mathbf{P}(J)^{-1} \right)(t).$$

Hence for $t \in I$:

$$\mathbf{E}((f - \mathbf{E}(f \mid \mathscr{F}_{j-1}))^2(\mathscr{F}_j))(t) \leqq \sum_{m \geqq m_0} a^2_{mi} \frac{1}{\mathbf{P}(I)} \int_I h^2_{\mathscr{G}_{mi}} + \sum_{m \geqq m_0} a_{mi} \frac{1}{\mathbf{P}(J)} \int_J h^2_{\mathscr{G}_{mi}}$$

$$\leqq 2 \sum_{(mi) \subset (m_0 j_0)} a^2_{mi} \cdot 4 \cdot 2^{m_0 - m}.$$

§2. Here we apply the information obtained above to the classification problem.

PROPOSITION 8. (a) *If*

$$\mathbf{P}\left(\bigcap_n^{\infty} \bigcup_{m=n}^{\infty} E_m \right) = 0 \quad \textit{and} \quad \sup_{B \in \mathscr{E}} \frac{1}{\mathbf{P}(B)} \sum_{E \in B \cap \mathscr{E}} \mathbf{P}(E) = \infty$$

then $H^1[(\mathscr{F}_n)]$ *is isomorphic to* $(\Sigma H^1_n)_{l^1}$.

(b) *If*

$$\mathbf{P}\left(\bigcap_n^{\infty} \bigcup_{m=n}^{\infty} E_m \right) > 0$$

then there exists a subspace of $H^1[(\mathscr{F}_n)]$ *which is isomorphic to* l^2.

PROOF. Fix $n \in \mathbf{N}$, define p as in Lemma 6. ad(a) $\delta_n := \inf\{\mathbf{P}(A) : A \in \mathscr{A}_n\}$. Fix $K_n > p \cdot n \cdot 8^n$. We inductively choose a sequence with the following properties: $m_0 =: 0$,

$$* \qquad \mathbf{P}\left(\bigcup_{m=m_n}^{\infty} E_n\right) < \tfrac{1}{8}\delta_{m_{n-1}}, \qquad n \geqq 1;$$

$$** \qquad \frac{1}{\mathbf{P}(B)} \sum_{j=m_{n-1}}^{m_n} \sum_{E\in B\cap \mathscr{E}_j} \mathbf{P}(E) \geqq K_n, \quad \text{for some } B \in \bigcup_{j=m_{n-1}}^{m_n} \mathscr{E}_j.$$

Take $f \in H^1[(\mathscr{F}_n)]$ we use Lemma 2 to obtain a minorization of $\| f \|_{H^1[(\mathscr{F}_n)]}$: Define $f_m := \mathbf{E}(f \mid \mathscr{F}_m) - E(f \mid \mathscr{F}_{m-1})$,

$$2\int S(f) \geqq \int \left(\sum_{n=1}^{\infty} \sum_{k=m_{2n}}^{m_{2n+1}} |f_k|^2\right)^{1/2} + \int \left(\sum_{n=1}^{\infty} \sum_{n=m_{2n-1}}^{m_{2n}} |f_k|^2\right)^{1/2}.$$

We minorize each integral separately (by using *, and Lemma 2):

$$c_n := \bigcup_{k=m_{2n}}^{m_{2n+1}} E_k \setminus \bigcup_{k=m_{2n+2}}^{\infty} E_k,$$

$$\int \left(\sum_{n=1}^{\infty} \left(\sum_{k=m_{2n}}^{m_{2n+1}} |f_k|^2\right) \chi_{c_n}\right)^{1/2} = \sum_{n=1}^{\infty} \int \left(\sum_{k=m_{2n}}^{m_{2n+1}} |f_k|^2\right)^{1/2} \cdot \chi_{c_n}$$

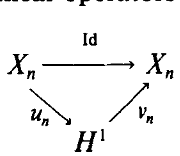
$$> \frac{7}{8} \sum_{n=1}^{\infty} \int \left(\sum_{k=m_{2n}}^{m_{2n+1}} |f_k|^2\right)^{1/2} \cdot \chi_{\cup_{k=m_{2n}}^{m_{2n+1}} E_k}$$

$$\geqq \frac{1}{4} \sum_{n=1}^{\infty} \int \left(S\left(\sum_{k=m_{2n}}^{m_{2n+1}} f_k\right)\right).$$

Moreover $X_n := (\operatorname{span}\{ f_m : f_m \in \mathscr{D}_m,\ m_n \leqq m < m_{m+1}\})$ contains a complemented copy of H_n^1 (by ** and Lemma 7).

All that implies that $H^1[(\mathscr{F}_n)]$ contains a complemented copy of $(\Sigma H_n^1)_{l^1}$.

On the other hand X_n is a 1-complemented subspace of $H^1[(\mathscr{F}_n)]$. By Maurey's theorem there exist linear operators u_n, v_n such that the diagram



commutes and $\| u_n \| \cdot \| v_n \| < c$ (with c independentof n). (Observe that we are actually factorizing through $H_{k_n}^1$ for some large k_n.)

Using the isomorphism $H^1[(\mathscr{F}_n)] \cong (\Sigma X_n)_{l^1}$ we conclude that the diagram

$$\begin{array}{ccc} H^1[(\mathscr{F}_n)] & \xrightarrow{\text{Id}} & H^1[(\mathscr{F}_n)] \\ & {\scriptstyle u}\searrow \quad \nearrow{\scriptstyle v} & \\ & (\Sigma H_n^1)_{l^1} & \end{array}$$

commutes, with $\| u \| \cdot \| v \| < \infty$. □

Now I apply the decomposition method, and we are done.

ad(b) We first choose $\mathscr{G} \subset \mathscr{E}$ such that

* for $l \in \mathbf{N}$, $E, F \in \mathscr{G} \cap \mathscr{A}_l$ we get $E \cap F \neq 0$ implies

$\operatorname{supp} h_E \cap \operatorname{supp} h_F \neq \varnothing$;

** for $\tilde{E}_n = (\mathscr{E}_n \cap \mathscr{G})^*$ we obtain $\mathbf{P}\left(\bigcap_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \tilde{E}_m \right) > 0.$

Next we observe that $\bigcap_j G_j^*(\Omega \mid \mathscr{G}) = \bigcap_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \tilde{E}_m$. Hence (by monotony) there exists $j_0 \in \mathbf{N}$ such that

$$\mathbf{P}\left(\bigcap_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \tilde{E}_m \right) \leqq \mathbf{P}(\mathscr{G}_j^*(\Omega \mid \mathscr{G})) \leqq 2\mathbf{P}\left(\bigcap_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \tilde{E}_m \right) \quad \text{for } j \geqq j_0.$$

By Lemma 5 there exists $\mathscr{B}_j \subset G_j(\Omega \mid \mathscr{G})$ such that:

$$S^2(h_{\mathscr{B}_j}) < \tfrac{3}{2} \quad \text{on } \Omega, \quad \text{and} \quad S^2(h_{\mathscr{B}_j}) > \tfrac{1}{2} \quad \text{on } G_j^*(\Omega \mid \mathscr{G});$$

moreover

$$\operatorname{supp} S^2(h_{\mathscr{B}_j}) \subset G_{j-1}^*(\Omega \mid \mathscr{G})$$

and

$$\sum_{j > j_0} a_j^2 S^2(h_{\mathscr{B}_j}) = S^2\left(\sum_{j > j_0} a_j h_{\mathscr{B}_j} \right) \quad \text{for } (a_j) \text{ arbitrary.}$$

It's now easy to see that $(h_{\mathscr{B}_j})_{j > j_0}$ is equivalent to the unit vector basis in l^2. Indeed,

$$\left\| \sum_{j > j_0} a_j h_{\mathscr{B}_j} \right\|_{H^1[(\mathscr{F}_n)]} = \int \left(\sum_{j > j_0} a_j^2 S^2(h_{\mathscr{B}_j}) \right)^{1/2}$$

and

$$\left(\sum_{j>j_0} a_j^2\right)^{1/2}\sqrt{\tfrac{1}{2}}\,\mathbf{P}\left(\bigcap\bigcap \hat{E}_m\right)\leqq \int\left(\sum a_j^2 S^2(h_{\mathscr{B}_j})\right)^{1/2}$$

$$\leqq\left(\sum_{j>j_0} a_j^2\right)^{1/2}\sqrt{\tfrac{3}{2}}\,\mathbf{P}(G_{j_0}^*(\Omega\mid\mathscr{G})).$$

By our choice of j_0:

$$\frac{\sqrt{3}}{2}\leqq\frac{\sqrt{\frac{3}{2}}}{\sqrt{2}}\frac{\mathbf{P}(G_{j_0}^*(\Omega\mid\mathscr{G}))}{\mathbf{P}\left(\bigcap_{n=1}^{\infty}\bigcap_{m=n}^{\infty}\tilde{E}_n\right)}\leqq\sqrt{3}.$$

PROPOSITION 9. *If*

$$\sup_{B\in\mathscr{E}}\frac{1}{\mathbf{P}(B)}\sum_{E\in B\cap\mathscr{E}}\mathbf{P}(E)<\infty,$$

then $H^1[(\mathscr{F}_n)]$ *is isomorphic to a complemented subspace of* l^1.

PROOF. Take $A\in\mathscr{A}_n$, n, A arbitrary,

$$M>\frac{1}{\mathbf{P}(A)}\sum_{E\in\mathscr{E}\cap A}\mathbf{P}(E)$$

$$=\frac{1}{\mathbf{P}(A)}\sum_{n\in\mathbf{N}}\sum_{E\in G_n(A)}\mathbf{P}(E)$$

$$=\frac{1}{\mathbf{P}(A)}\sum_{n\in\mathbf{N}}\mathbf{P}(G_n^*(A)).$$

Hence $\mathbf{P}(G_{4M}^*(A))\leqq\mathbf{P}(A)/4$ (cf. [2], p. 820).

Given $f=\sum h_A a_A$ with $f\in H^1[(\mathscr{F}_n)]$ we write $G_n:=G_n(\Omega)^*$:

$$\|S(f)\|_1=\int\left(\sum_{n\in\mathbf{N}}S^2\left(\sum_{A\in G_n(\Omega)}h_A a_A\right)\right)^{1/2}$$

$$\geqq\frac{1}{4M}\sum_{j=1}^{4M}\int\left(\sum_{n\in\mathbf{N}}S^2\left(\sum_{A\in G_{4Mn+j}(\Omega)}h_A a_A\right)\right)^{1/2}$$

$$\geqq\frac{1}{4M}\sum_{j=1}^{4M}\sum_{n\in\mathbf{N}}\int S\left(\sum_{A\in G_{4Mn+j}(\Omega)}h_A a_A\right)\chi_{G_{4Mn+j}\setminus\bigcup_{m=n+1}^{\infty}G_{4Mm+j}}$$

$$\geqq \frac{1}{8M} \sum_{j=1}^{4M} \sum_{n \in \mathbf{N}} \int S\left(\sum_{A \in G_{4Mn+j}(\Omega)} h_A a_A \right) \chi_{G_{4Mn+j}}.$$

Fix now $n \in \mathbf{N}$:

$$\int S\left(\sum \{h_A a_A : A \in G_n(\Omega)\} \right) \chi_{G_n}$$

$$= \sum_l \int S\left(\sum \{h_A a_A : A \in G_n(\Omega)\} \right) \chi_{(G_n(\Omega) \cap \mathscr{A}_l)^*}$$

$$\geqq \frac{1}{2} \sum_l \int S\left(\sum \{h_A a_A : A \in G_n(\Omega) \cap \mathscr{A}_l\} \right).$$

Define now

$$X_{n,l} := (\{\sum h_A a_A : A \in G_n(\Omega) \cap \mathscr{A}_l\}, \| \quad \|_{H^1}).$$

We have shown up to now that

$$H^1[(\mathscr{F}_n)] \text{ is isomorphic to } \left(\sum_{n,l} X_{n,l} \right)_{l^1}.$$

It remains to show that $X_{n,l}$ is uniformly complemented in l^1. To do so, we observe that

$$i_{n,l} : X_{n,l} \to l^1,$$

$$f \to ((f/B) \cdot \mathbf{P}(B), B \in G_n(\Omega) \cap \mathscr{A}_l)$$

is an isomorphism (by Lemma 2(b)).

Moreover, by Lemma 2(a), for any sequence β_B, $B \in G_n(\Omega) \cap \mathscr{A}_l$ there exists a well-defined sequence (a_A) such that for

$$f = \sum \{h_A a_A, A \in G_n(\Omega) \cap \mathscr{A}_l\}$$

we get

$$\beta_B = f/B \cdot \mathbf{P}(B).$$

Hence there exists $P_{n,l} : l^1 \to X_{n,l}$ such that

$$P_{n,l} i_{n,l} = \mathrm{id}_{X_{n,l}} \text{ and } \quad \| P_{n,l} \| \cdot \| i_{n,l} \| \leqq C.$$

PROOF OF THEOREM 1, PART(b). Proposition 9 and Proposition 8(b) imply that the hypothesis of Proposition 8(a) is satisfies. Hence $H^1[(\mathscr{F}_n)]$ is isomorphic to $(\Sigma H_n^1)_{l^1}$.

PROOF OF THEOREM 1(a).

Part a. This is Theorem B (cf. [5]).

Part b. Combine Theorem B(b) and Proposition 8(b) to see that the hypothesis of Proposition 8(a) is satisfied.

Part c. Combine Proposition 9 with the fact that any complemented subspace of l^1 is isomorphic to l^1 (cf. [3]).

REFERENCES

1. J. B. Garnett, *Bounded Analytic Functions*, Academic Press, 1981.

2. P. W. Jones, *BMO and the Banach Space Approximation problem*, Amer. J. Math. **107** (1985), 853–893.

3. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, 1977.

4. B. Maurey, *Isomorphism entre Espaces H^1*, Acta Math. **145** (1980), 79–120.

5. P. F. X. Müller, *On subsequences of the Haar basis in $H^1(\delta)$ and isomorphism between H^1 spaces*, Studia Math. to appear.

6. P. F. X. Müller, *On projections in H^1 and BMO*, preprint.

7. P. F. X. Müller, *On the span of some three valued martingale difference sequences in L^p and H^1*, in preparation.